The Decoupled Polynomial NARX Model: Parameter Reduction and Structural Insights for the Bouc-Wen Benchmark

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The Nonlinear Autoregressive Exogenous Input Model

$$
\begin{aligned}
y(t) &= f(\boldsymbol{z}(t)) + e(t) = \sum_{k=1}^{K} \gamma_k \phi_k(\boldsymbol{z}(t)) + e(t) \\
\boldsymbol{z}(t) &= \begin{bmatrix} u(t) & \dots & u(t-n_b) & y(t-1) & \dots & y(t-n_a) \end{bmatrix}^T
\end{aligned}
$$

Pros

- Linear in Parameters
- Generality

Cons

- Many, many parameters
- Difficult to Interpret
- Lack of a Noise Model

Applications of NARX Models

There have been many recent applications of NARX modelling, *with no direct connection to the identification community*.

- Analysis of EEG
- Model of biogas producing waste digestor
- Electric Power Load Forecasting
- Wind Speed and Power Forecasting
- Hysteresis Modelling in Passive Building Control Systems

Polynomial NARX Models

$$y(t) = \sum_{k=1}^{K} \gamma_k \phi_k(\boldsymbol{z}(t)) + e(t)$$

- Many choices possible for basis elements: $\phi_k(\boldsymbol{z})$
  - Polynomials, Sigmoids, Radial Basis Functions, etc....
- Polynomials are commonly used,

Cons

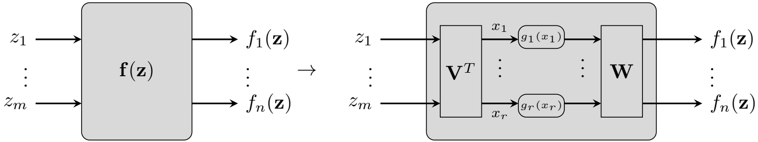
- Poor extrapolation
- Can lead to instability in simulation

Pros

- Connection to Volterra theory

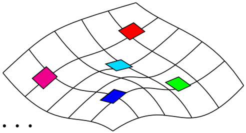
MIMO Polynomial Decoupling

- Decoupling a polynomial can reveal structure.
- Applications in Parallel Wiener-Hammerstein models, Polynomial State-Space models



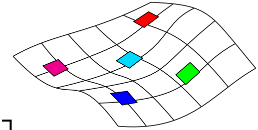
Jacobian-Based Polynomial Decoupling

- Probe $\mathbf{f}(\mathbf{z})$ at sampling points
- Collect Jacobian matrices



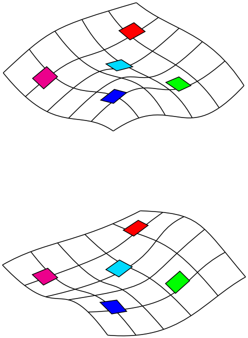
$$\mathbf{J_f}(\mathbf{z}^{(1)}), \mathbf{J_f}(\mathbf{z}^{(2)}), \mathbf{J_f}(\mathbf{z}^{(3)}), \mathbf{J_f}(\mathbf{z}^{(4)}), \mathbf{J_f}(\mathbf{z}^{(5)}), \dots$$

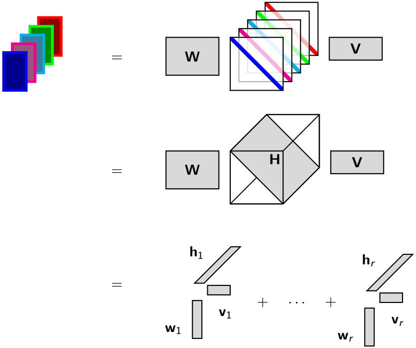
- Jacobian of $\mathbf{f}(\mathbf{z})$



$$\mathbf{f}(\mathbf{z}) = \mathbf{W}\left[g_i(\mathbf{v}_i^T\mathbf{z})\right]$$
$$\Downarrow$$
$$\mathbf{J_f}(\mathbf{z}) = \mathbf{W}\begin{bmatrix} g_1'(\mathbf{v}_1^T\mathbf{z}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & g_r'(\mathbf{v}_r^T\mathbf{z}) \end{bmatrix}\mathbf{V}^T$$

Tensor of Stacked Jacobian Matrices





Single-Output Case

- Single output, each Jacobian is now a vector, not a matrix.
- Stacking operating points creates a matrix.

$$\boldsymbol{J} = \boldsymbol{V}\boldsymbol{H}^T$$

- Matrix factorization problem
- Tensor uniqueness results no longer apply!

Enforcing Polynomial Structure on $\boldsymbol{H}$

$$
\begin{aligned}
x_k(t) &= \boldsymbol{z}^T(t)\boldsymbol{v}_k \\
g_k'(x_k(t)) &= \sum_{\ell=0}^{n-1} c_{k,\ell}\, x_k^{\ell}(t) \\
\boldsymbol{X}_k &= \begin{bmatrix} 1 & x_k(1) & x_k^2(1) & \dots & x_k^{n-1}(1) \\ 1 & x_k(2) & x_k^2(2) & \dots & x_k^{n-1}(2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_k(N) & x_k^2(N) & \dots & x_k^{n-1}(N) \end{bmatrix} \\
\boldsymbol{H}(\boldsymbol{V},\boldsymbol{C}) &= \begin{bmatrix} \boldsymbol{X}_1\boldsymbol{c}_1 & \boldsymbol{X}_2\boldsymbol{c}_2 & \dots & \boldsymbol{X}_r\boldsymbol{c}_r \end{bmatrix}
\end{aligned}
$$

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Decoupled NARX Algorithm

1. Fit a polynomial NARX model between $u(t)$ and $y(t)$

$$\hat{y}(t|t-1) = f(\mathbf{z}(t))$$

2. Compute the Jacobian

$$J(i,t) = \frac{\partial f(\mathbf{z}(t))}{\partial z_i(t)}$$

3. Factor the Jacobian using the optimization

$$(\hat{\mathbf{V}}, \hat{\mathbf{C}}) = \min_{\mathbf{V},\mathbf{C}} \left\| \mathbf{J} - \mathbf{V}\mathbf{H}(\mathbf{V},\mathbf{C}) \right\|_F$$

Final Optimization

- Optimization on Identification Data

$$\begin{aligned}\hat{\mathbf{y}}(\mathbf{V},\mathbf{C}) &= \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \dots & \mathbf{X}_r \end{bmatrix} \mathrm{vec}(\mathbf{C}) \\ (\hat{\mathbf{V}},\hat{\mathbf{C}}) &= \min_{\mathbf{V},\mathbf{C}} \|\mathbf{y} - \hat{\mathbf{y}}(\mathbf{V},\mathbf{C})\|_2\end{aligned}$$

- Linear in polynomial coeffiients $\mathbf{C}$.
- Nonlinear in mixing matrix, $\mathbf{V}$.
- Remove redundancies (constants, gains).
- Similarity to Wiener model identification.

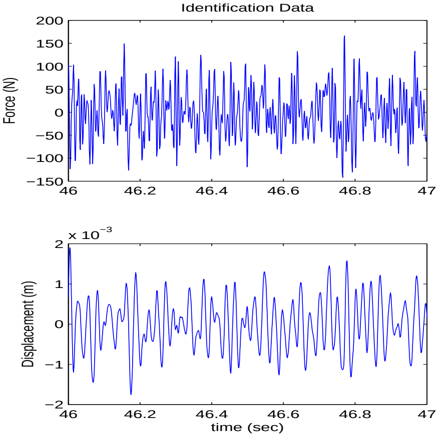
Bouc-Wen Benchmark

- Hysteresis model
- Continuous time state-space relization:

$$\begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{K_L}{m_L} & -\frac{c_L}{m_L} & -\frac{1}{m_L} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_L} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\beta\gamma & -\beta\delta \end{bmatrix} \begin{bmatrix} |\dot{y}(t)|z(t) \\ \dot{y}(t)|z(t)| \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \\ z(t) \end{bmatrix}$$

Identification Data



- Data generation using example provided with Benchmark
- 5 periods of 8192 samples of multisine, 50 N RMS.
- 750 Hz sampling, 54.6 seconds of data
- SNR 40dB

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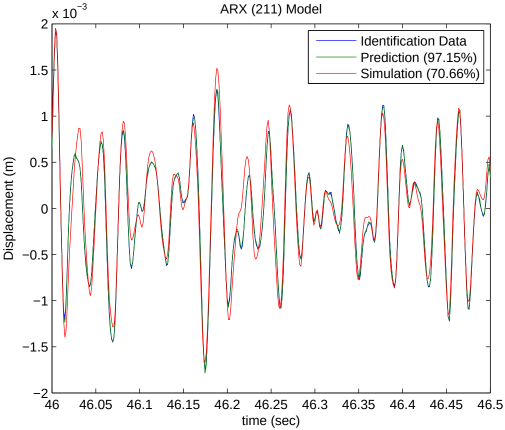
Linear ARX models



- Minimizing AIC or MDL suggests $na = 2$, $nb = 1$, $nk = 1$.
- Prediction: 97.15%
- Simulation: 70.66%

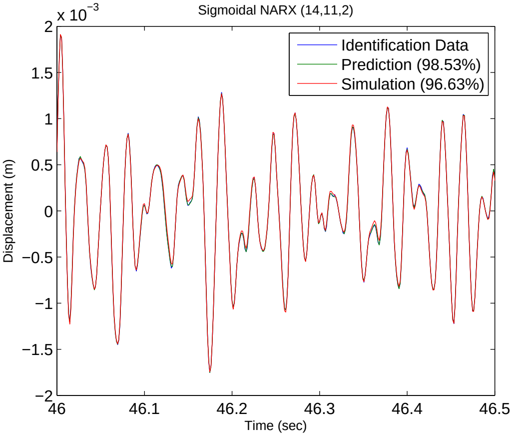
Standard NARX models

- Nonlinearities: Wavenet and Sigmoid
- Scan over
  - past outputs: $1 \leq n_a \leq 15$
  - past inputs: $1 \leq n_b \leq 15$
  - input delay: $0 \leq n_k \leq 5$
- Minimize AIC to select structure

$$n_a = 14, n_b = 11, n_k = 2$$

- Probably better to split data, and cross-validate.

Sigmoidal NARX model

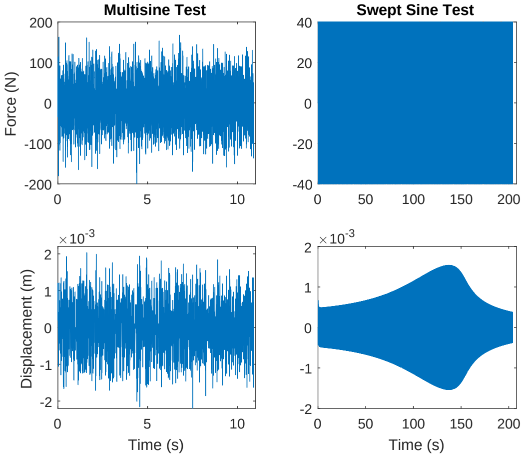


- Excellent performance in both Prediction and Simulation
- 1571 parameters!

NARX Results (estimation data)

NARX Model Type	Structure	# Param.	Pred.	Sim.
Linear	(2,1,1)	3	97.2%	70.7%
Sigmoidal 10 units	(14,11,2)	1571	98.5%	96.6%
Full Degree 3 Polynomial	(14,11,2)	3250	98.6%	97.0%
Pruned Degree 3 Polynomial	(14,11,2)	568	98.6%	96.7%
Decoupled 5 Branch Deg. 3	(14,11,2)	141	98.3%	94.6%
Decoupled 5 Branch Deg. 5	(14,11,2)	151	98.4%	95.2%
Decoupled 5 Branch Deg. 7	(14,11,2)	161	98.5%	96.1%

Validation Data



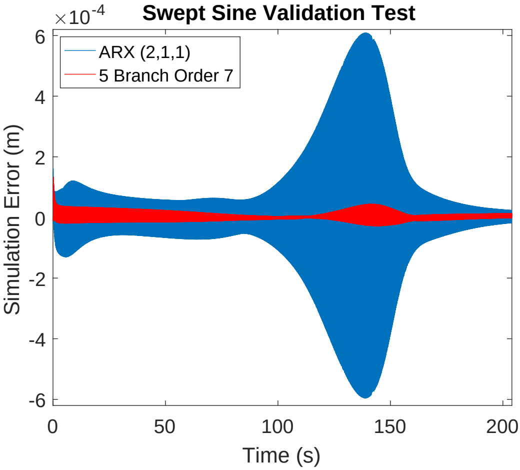
- Noise-Free Validation Signals
- Figure of Merit: Unnormalized RMS Error
- One-Step Ahead Prediction and Pure Simulation

NARX Results (validation data)

Simulation RMSE for two validation signals

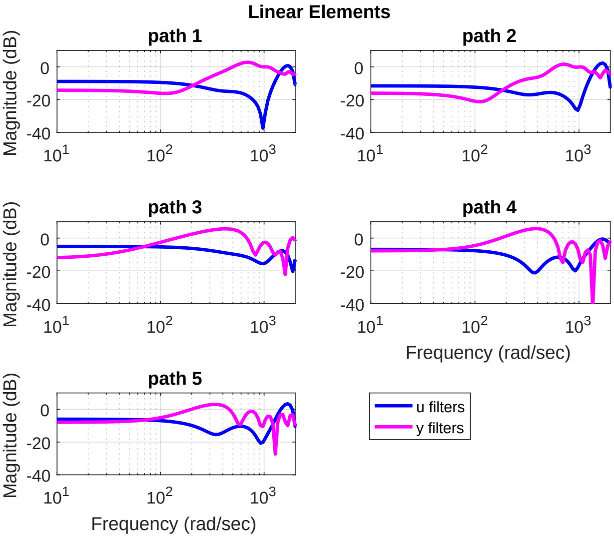
NARX Model Type	MultiSine $\times 10^{-4}$	Swept Sine $\times 10^{-4}$
Linear	1.71	1.61
Sigmoidal 10 units	0.53	0.15
Decoupled 5 Branch Deg. 3	0.59	0.21
Decoupled 5 Branch Deg. 5	0.57	0.19
Decoupled 5 Branch Deg. 7	∞	0.14

Swept Sine Test

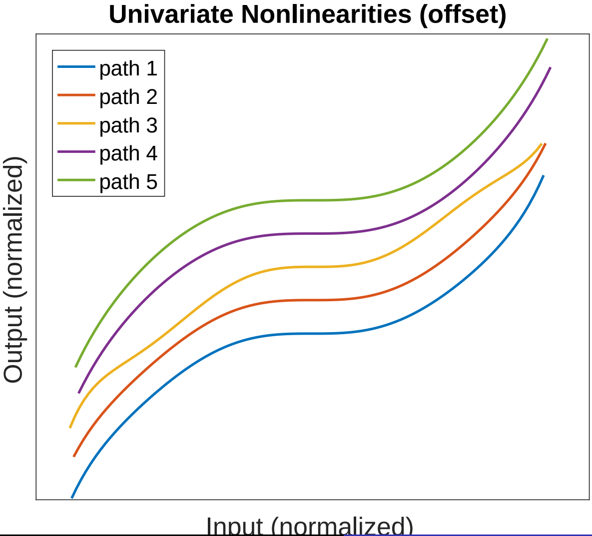


- Linear Model: 3 parameters, RMS error: -75.9dB
- Decoupled Model: 161 parameters, RMS error: -97.1 dB

Linear Elements



Nonlinearities



Conclusions

- Reduction in number of model parameters
- Insight from Linear Elements
- Simplicity of dealing with Univariate Nonlinearities.
- Poor extrapolation can lead to instability in simulation!