Phase-based homogeneous order separation for improving Volterra series identification

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Introduction

Volterra series and order separation

Phase-based homogeneous order separation

Evaluation and results

**System Identification**

$$u(t) \rightarrow y(t) \equiv u(t) + \sum_{i=1}^{N} h_i y_i(t)$$

- **Linear subsystem**: $y_1(t)$
- **Quadratic subsystem**: $y_2(t)$
- **Cubic subsystem**: $y_3(t)$

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\[ u(t) \rightarrow \text{NL system} \rightarrow y(t) \equiv u(t) \rightarrow \text{Identification} \rightarrow \{h_1, h_2, \ldots, h_N\} \]

Direct identification

System \ \{u\} \rightarrow \text{Order separation} \rightarrow \{y\} \rightarrow \text{Identification} \rightarrow \{h_1, h_2, \ldots, h_N\} \rightarrow h_1, \ldots, h_N

Identification on separated orders
Introduction

Volterra series and order separation
- Recalls on Volterra series
- Order separation using amplitude gains

Phase-based homogeneous order separation
- Theoretical method for complex-valued signals
- Extension for real-valued signals

Evaluation and results
- Evaluation of order separation on a simulated system
- Application to the Silverbox benchmark
Recalls on Volterra series¹

\[ y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n_+} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t - \tau_i) \, d\tau_i \]

Recalls on Volterra series

\[ y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n_+} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t - \tau_i) d\tau_i \]

\[ = \sum_{n=1}^{+\infty} V_n[u, \ldots, u](t) \]

Properties of operator \( V_n \):

- **Symmetry** (considering \( h_n \) symmetric)
  \[ V_n[u_1, \ldots, u_n] = V_n[u_{\pi(1)}, \ldots, u_{\pi(n)}], \forall \text{ permutations } \pi \]

- **Multilinearity**
  \[ V_n[u_1, \ldots, \lambda u_k + \mu v, \ldots, u_n] = \lambda V_n[u_1, \ldots, u_k, \ldots, u_n] + \mu V_n[u_1, \ldots, v, \ldots, u_n] \]

- **Homogeneity**
  \[ V_n[\alpha u_1, \ldots, \alpha u_n] = \alpha^n V_n[u_1, \ldots, u_n] \]

---

Order separation using amplitude gains\(^2\)

\[ \alpha u(t) \xrightarrow{S} z(t) = \begin{bmatrix} \alpha & \alpha^2 & \ldots & \alpha^N \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}(t) \]

Order separation using amplitude gains\(^2\)

\[
\begin{bmatrix}
\alpha_1 u \\
\alpha_2 u \\
\vdots \\
\alpha_K u \\
\end{bmatrix}(t) \rightarrow S \quad \rightarrow \\
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_K \\
\end{bmatrix}(t) = \\
\begin{bmatrix}
\alpha_1 & \alpha_2^2 & \cdots & \alpha_N^3 \\
\alpha_2 & \alpha_2^2 & \cdots & \alpha_N^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_K & \alpha_K^2 & \cdots & \alpha_K^N \\
\end{bmatrix} \\
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_N \\
\end{bmatrix}(t)
\]

Vandermonde matrix

Advantages and disadvantages

- Easy implementation
- Bad conditioning of Vandermonde matrix when \( N \) is large ⟹ sensibility to noise
- Difficulties in choosing the gains \( \alpha_k \):
  - \( \alpha_k > 1 \) ⟹ potential saturation of the system
  - \( \alpha_k < 1 \) ⟹ higher-orders hidden in noise

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Hypothesis

Input signal $u(t) \in \mathbb{C}$
Hypothesis

Input signal $u(t) \in \mathbb{C}$

$$
\begin{bmatrix}
  u \\
  wu \\
  \vdots \\
  w^{N-1}u
\end{bmatrix}
(t) \rightarrow
S
\rightarrow
\begin{bmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_N
\end{bmatrix}
(t)
= \begin{bmatrix}
  1 & 1 & \ldots & 1 \\
  w & w^2 & \ldots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  w^{N-1} & w^{2N-2} & \ldots & 1
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_N
\end{bmatrix}
(t)
$$

with $w = e^{j2\pi/N}$

Discrete Fourier Transform (DFT) matrix of order $N$
Hypothesis

Input signal $u(t) \in \mathbb{C}$

\[
\begin{bmatrix}
  u \\
  wu \\
  \vdots \\
  w^{N-1}u
\end{bmatrix}(t) \xrightarrow{S} \begin{bmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_N
\end{bmatrix}(t) = \begin{bmatrix}
  1 & 1 & \ldots & 1 \\
  w & w^2 & \ldots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  w^{N-1} & w^{2N-2} & \ldots & 1
\end{bmatrix} \begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_N
\end{bmatrix}(t)
\]

Discrete Fourier Transform (DFT) matrix of order $N$

with $w = e^{j2\pi/N}$
Hypothesis

Input signal $u(t) \in \mathbb{C}$

$$\begin{bmatrix} u \\ wu \\ \vdots \\ w^{N-1}u \end{bmatrix}(t) \xrightarrow{S} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}(t) = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ w & w^2 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ w^{N-1} & w^2 & \ldots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}(t)$$

Discrete Fourier Transform (DFT) matrix of order $N$

with $w = e^{j2\pi/N}$

Advantages and disadvantages

- **Optimal conditioning**
- **Reduces measurement noise by a factor $\sqrt{N}$ (supposing Gaussian white noise)**
- **Predictable behaviour if wrong truncation order $N$**
- **Need complex signals as input & output ∼ theoretical method**
Approach for real-valued signals

Choice of input signal: $\text{Re}[wu(t)]$, with $w = e^{j2\pi/N}$

$$y_1 = V_1 \left[ w u + w^{-1} \bar{u} \right]$$
Approach for real-valued signals

Choice of input signal: $\text{Re}[wu(t)]$, with $w = e^{j2\pi/N}$

Notation: intermodulation term

$$V_{n,q}(t) = V_n\left[u, \ldots, u, \bar{u}, \ldots, \bar{u}\right](t)$$

$$n-q \text{ times } \quad q \text{ times}$$

$$y_1 = w^{-1}V_{1,1} + wV_{1,0}$$
Approach for real-valued signals

**Choice of input signal:** Re\[wu(t)\], with \( w = e^{j2\pi/N} \)

**Notation:** intermodulation term

\[
V_{n,q}(t) = V_n\left[u, \ldots, u, \bar{u}, \ldots, \bar{u}\right](t)
\]

\[
V_{n,q}(t) = V_n\left[u, \ldots, u, \bar{u}, \ldots, \bar{u}\right](t)
\]

\[n - q \text{ times} \quad q \text{ times}\]

\[
y_1 = w^{-1}V_{1,1} + wV_{1,0}
\]

\[
y_2 = w^{-2}V_{2,2} + 2V_{2,1} + w^2V_{2,0}
\]
Approach for real-valued signals

Choice of input signal: \( \Re[wu(t)] \), with \( w = e^{j2\pi/N} \)

Notation: *intermodulation term*

\[
V_{n,q}(t) = V_n \left[ u, \ldots, u, \bar{u}, \ldots, \bar{u} \right] (t)
\]

\[n-q\] times \[q\] times

\[y_1 = w^{-1} V_{1,1} + w V_{1,0}\]

\[y_2 = w^{-2} V_{2,2} + 2 V_{2,1} + w^2 V_{2,0}\]

\[y_3 = w^{-3} V_{3,3} + 3 w^{-1} V_{3,2} + 3 w V_{3,1} + w^3 V_{3,0}\]
\[
\begin{bmatrix}
\text{Re}[u] \\
\text{Re}[wu] \\
\vdots \\
\text{Re}[w^{2N}u]
\end{bmatrix}(t) \xrightarrow{S} 
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_{2N+1}
\end{bmatrix}(t) = 
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
w & w^2 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
w^{2N} & w^{4N} & \ldots & 1
\end{bmatrix}
\begin{bmatrix}
2V_{2,1} \\
V_{1,0} + 3V_{3,1} \\
V_{2,0} \\
V_{3,0} \\
V_{3,3} \\
V_{2,2} \\
V_{1,1} + 3V_{3,2}
\end{bmatrix}(t)
\]

with \( w = e^{j2\pi/(2N+1)} \)

Discrete Fourier Transform (DFT) matrix of order \( 2N + 1 \)
Input signal: $\text{Re}[(w_1 + w_2)u(t)]$
\[ \{ u_{k_1,k_2} = \text{Re}[(w^{k_1} + w^{k_2})u] \} \]
\{ u_{k_1,k_2} = \text{Re}\left[ (w^{k_1} + w^{k_2})u \right] \} \\

\{ z_{k_1,k_2} \} \\

\{ V_{n,q} \} \\

\begin{array}{|c|}
\hline
\text{Separation method} \\
\hline
\text{Amplitude-based} & \text{Phase-based} \\
\hline
\text{Signal type} & \text{real-valued} & \text{complex-valued} & \text{real-valued} \\
\text{Method parameters} & \text{Gains } \alpha_k & \text{None} & \text{None} \\
\text{Number of measurements} & \text{N} & \text{N} & \text{(2N+1)(N+1)} \\
\text{Conditioning} & \text{Bad} & \text{Optimal} & \text{Good} \\
\text{Noise reduction} & \checkmark & \checkmark & \checkmark \\
\text{Order rejection} & & & \\
\hline
\end{array}
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## Order separation evaluation test

### Test description

**Simulated system:**
- Kernel with memory length of 5 samples
- Truncation order $N = 9$
- In total, 2001 parameters to estimate

**Tested separation methods:**
- Phase-based, with 190 test signals
- Amplitude-based, with 9 test signals
- Amplitude-based, with 190 test signals

**Input signal:** Gaussian white noise

**Kernel identification:** Least-Squares method (time-domain)
Order separation error

![Graph showing order separation error](image-url)

- **Amp. method**
- **Amp. method (2)**
- **Phase method**
- **Noise level**

<table>
<thead>
<tr>
<th>Order n</th>
<th>NRMSE (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-80</td>
</tr>
<tr>
<td>2</td>
<td>-60</td>
</tr>
<tr>
<td>3</td>
<td>-40</td>
</tr>
<tr>
<td>4</td>
<td>-20</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>7</td>
<td>40</td>
</tr>
<tr>
<td>8</td>
<td>60</td>
</tr>
<tr>
<td>9</td>
<td>80</td>
</tr>
</tbody>
</table>

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Kernel identification error

![Graph showing kernel identification error with NRMSE (dB) vs. Order n. The graph compares 'Direct' and 'True orders.' The 'Direct' line shows a higher NRMSE, while the 'True orders' line shows a lower NRMSE.](image-url)
Kernel identification error

![Graph showing NRMSE (dB) vs Order n for different methods: Direct, True orders, Amp. method, Amp. method (2), and Phase method. The graph plots NRMSE on the y-axis and Order n on the x-axis.]
Silverbox benchmark

Test description

- Simulation of the Silverbox model
- Separate orders of the test data using the phase-based method
- Suppose truncation order $N = 9$ ($\approx 10$ hours of measurement needed)
Silverbox benchmark: spectra of separated orders
Silverbox benchmark: spectra of separated orders

![Graph showing spectra of separated orders](image-url)
Homogeneous order separation for identification

✓ Modular approach: any identification method can be used
✓ Applicable to any polynomial series expansion
  • Volterra series
  • Block systems with polynomial nonlinearities (Wiener, Hammerstein, Wiener-Hammerstein)
  • Polynomial NARMAX
  • Polynomial Nonlinear State-Space (PNLSS)

✗ Need a great number of measurements (i.e. \((2N + 1)(N + 1) \equiv O(N^2)\))

Conclusion

Homogeneous order separation for identification

- Modular approach: any identification method can be used
- Applicable to any polynomial series expansion
  - Volterra series
  - Block systems with polynomial nonlinearities (Wiener, Hammerstein, Wiener-Hammerstein)
  - Polynomial NARMAX
  - Polynomial Nonlinear State-Space (PNLSS)
- Need a great number of measurements (i.e. \((2N + 1)(N + 1) \equiv O(N^2)\))


Thanks for your attention!
Writing PNLSS output as sum of homogeneous order

\[ \dot{x}(t) = Ax(t) + Bu(t) + \sum_{(p,q) \in \mathbb{N}^2} M_{pq} \left( x(t), \ldots, x(t), u(t), \ldots, u(t) \right) \]

With \( W(s) = [sI - A]^{-1} \)