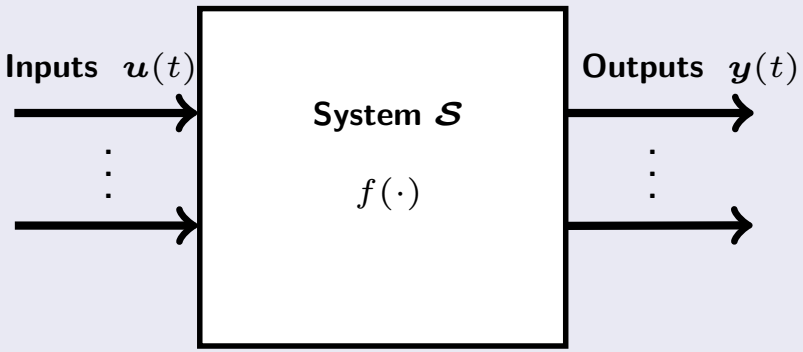
Lifting the curse of dimensionality in nonlinear system identification with tensor networks.

Kim Batselier, Zhongming Chen, Ching-Yun Ko, Ngai Wong

System identification



Main Problem

Given measured inputs $\{\boldsymbol{u}(t) \in \mathbb{R}^p\}_{t=1}^N$, outputs $\{\boldsymbol{y}(t) \in \mathbb{R}^l\}_{t=1}^N$, and a parametric input-output mapping $f(\cdot)$ for the system $\boldsymbol{\mathcal{S}}$, estimate the parameters of $f(\cdot)$.

NFIR black box model

- $\boldsymbol{y}(t) = f(\boldsymbol{u}(t), \boldsymbol{u}(t-1), \ldots, \boldsymbol{u}(t-M+1))$ with nonlinear function $f(\cdot)$

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- SISO truncated Volterra series:

$$y(t) = h_0 + \sum_{i=1}^{d} \sum_{k_1,\ldots,k_i=0}^{M-1} h_i(k_1, \ldots, k_i) \prod_{j=1}^{i} u(t-k_j),$$

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- MIMO truncated Volterra series: each of the l outputs in $\boldsymbol{y}(t) \in \mathbb{R}^l$ is a multivariate polynomial in $\boldsymbol{u}(t), \boldsymbol{u}(t-1), \ldots, \boldsymbol{u}(t-M+1)$.

Multivariate polynomials

- n-variate polynomial of total degree d
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Truncated Volterra series are described by an exponential amount of unknown coefficients $\Rightarrow$ need to estimate an exponential amount of coefficients.

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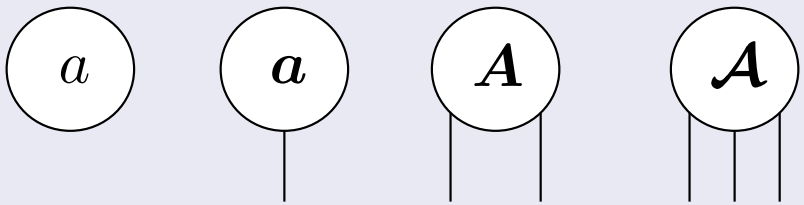
Main message of this talk

Lifting the curse of dimensionality in this system identification problem with tensor networks.

Tensor network building blocks



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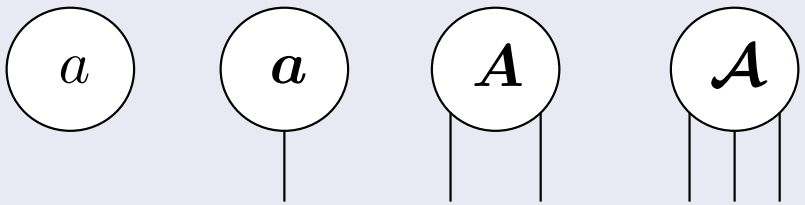

Three tensor network diagram rules

1. Nodes in the network are tensors
2. Each edge of each node is a particular index of the tensor
3. Connected edges refer to summation over that particular index

Tensor network example 1

$$\boldsymbol{C}(i,j) = \sum_k \boldsymbol{A}(i,k)\ \boldsymbol{B}(k,j)$$

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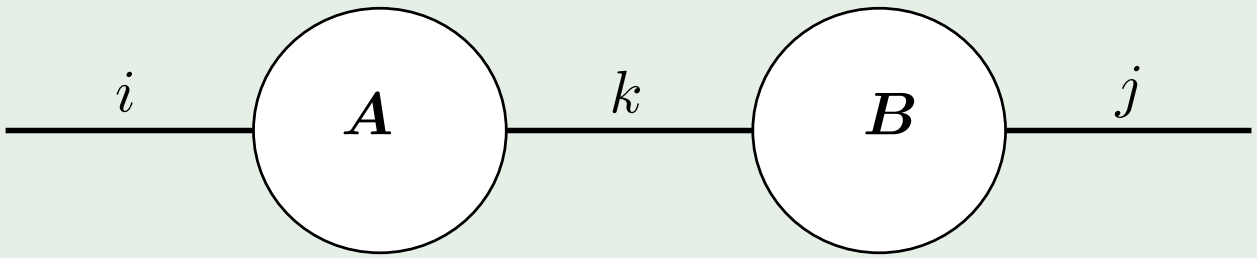

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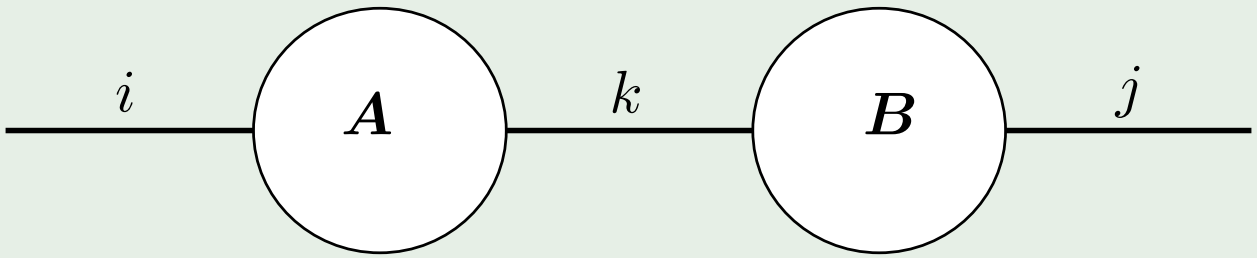
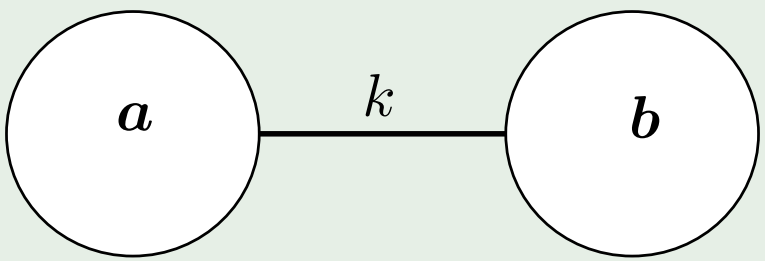


Tensor network example 2



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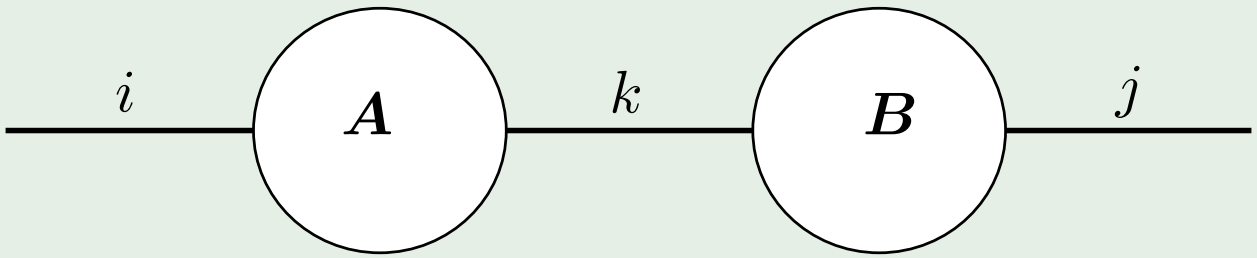
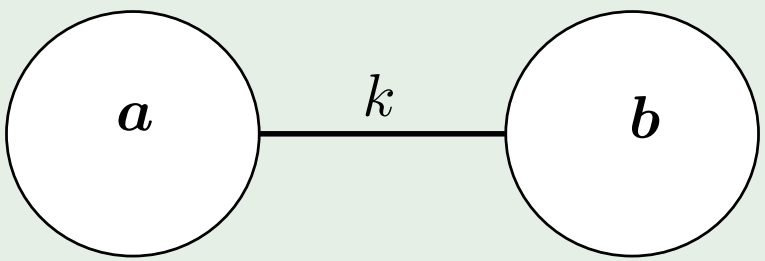
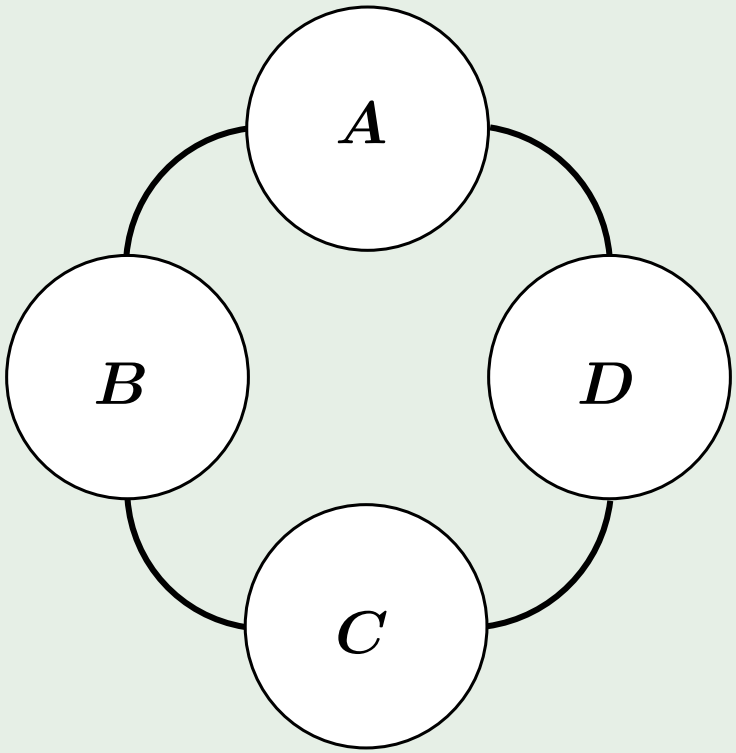


Tensor network example 2



$\sum_k \boldsymbol{a}(k)\ \boldsymbol{b}(k) = \boldsymbol{a}^T\ \boldsymbol{b}$

Tensor network example 3



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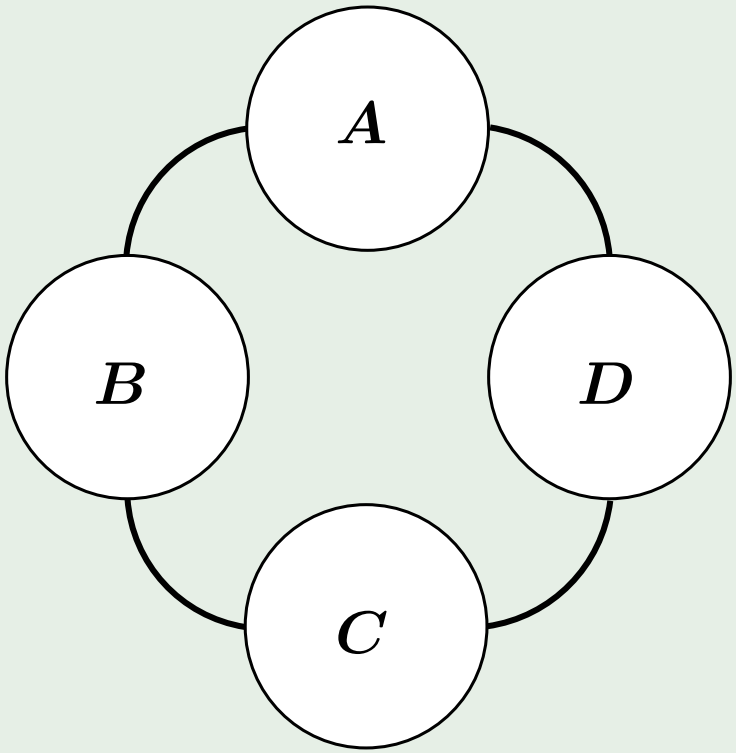

$$\text{Trace}(\boldsymbol{ABCD}) = \text{Trace}(\boldsymbol{BCDA}) = \cdots = \text{Trace}(\boldsymbol{DABC})$$

Tensor network example 4

Singular value decomposition of a rank-r matrix $\boldsymbol{A} \in \mathbb{R}^{p \times q}$:

$$\boldsymbol{A} = \boldsymbol{U}\ \boldsymbol{S}\ \boldsymbol{V}^T$$

with $\boldsymbol{U} \in \mathbb{R}^{p \times r}, \boldsymbol{V} \in \mathbb{R}^{q \times r}$ orthogonal and $\boldsymbol{S} \in \mathbb{R}^{r \times r}$ diagonal.

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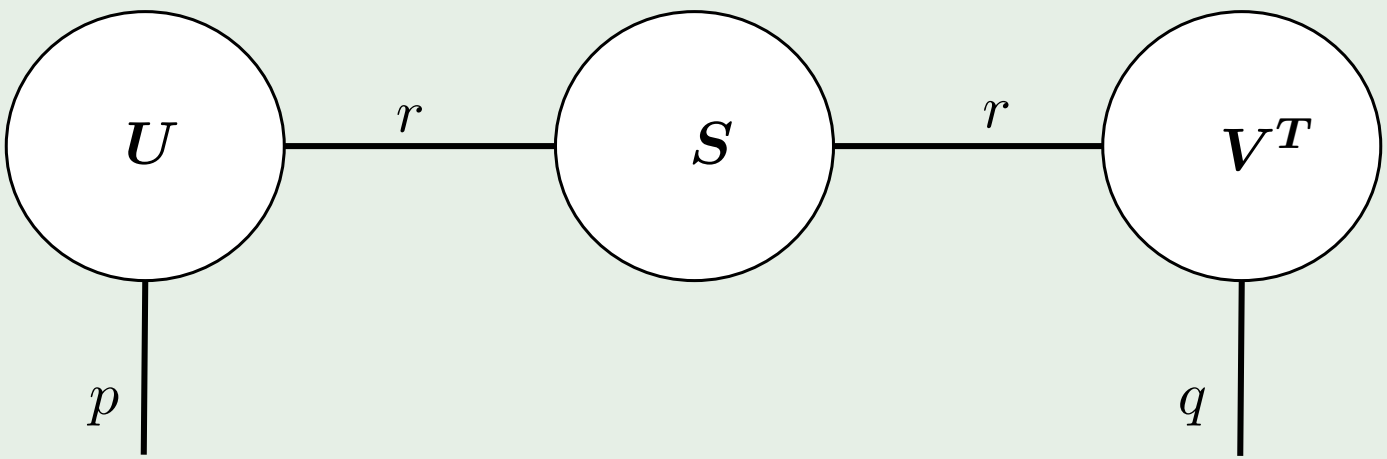

Outer product

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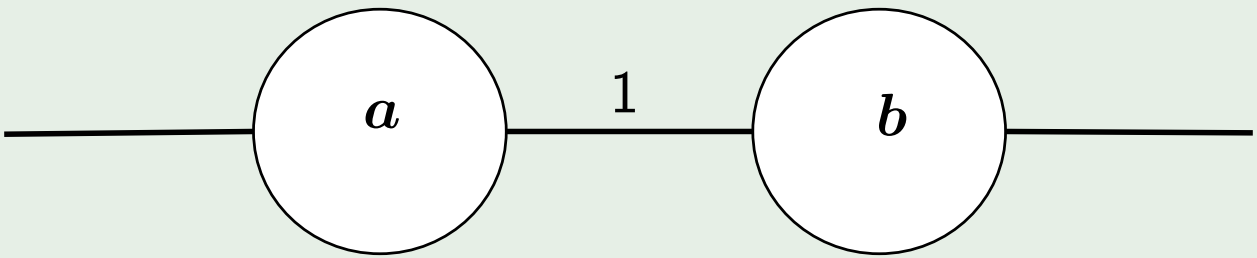

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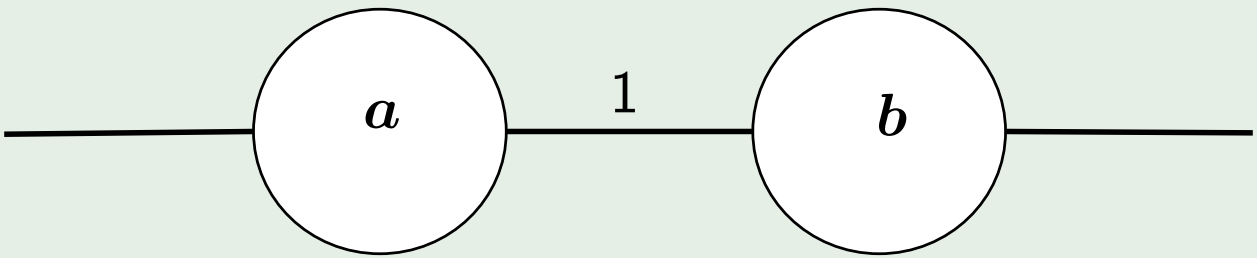

Important take-home message

Tensor networks with small interconnection dimensions represent tensors with interrelated entries.

MIMO Volterra model

Each of the l outputs in $\boldsymbol{y}(t) \in \mathbb{R}^l$ is a multivariate polynomial in $\boldsymbol{u}(t), \boldsymbol{u}(t-1), \ldots, \boldsymbol{u}(t-M+1)$.

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Define input and output vectors

- $\boldsymbol{y}(t) := \begin{pmatrix} y_1(t) \\ \vdots \\ y_l(t) \end{pmatrix} \in \mathbb{R}^l$
- $\boldsymbol{u}_t := \begin{pmatrix} 1 & \boldsymbol{u}(t)^T & \cdots & \boldsymbol{u}(t-M+1)^T \end{pmatrix}^T \in \mathbb{R}^{(pM+1)}$
- All input monomials are contained in

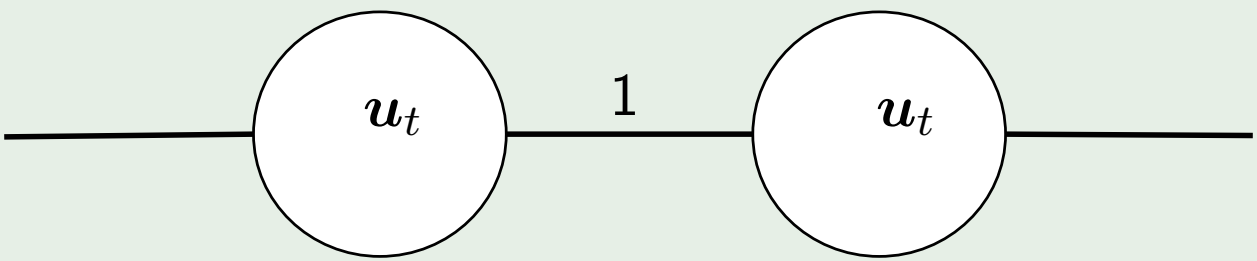
$$\overbrace{\boldsymbol{u}_t \circ \boldsymbol{u}_t \circ \cdots \circ \boldsymbol{u}_t}^{d} \in \mathbb{R}^{(pM+1)\times\cdots\times(pM+1)}$$

Small example: $p = 1$, $l = 1$, $M = 2$ and $d = 2$

$$\boldsymbol{u}_t = \begin{pmatrix} 1 \\ u(t) \\ u(t-1) \end{pmatrix}$$

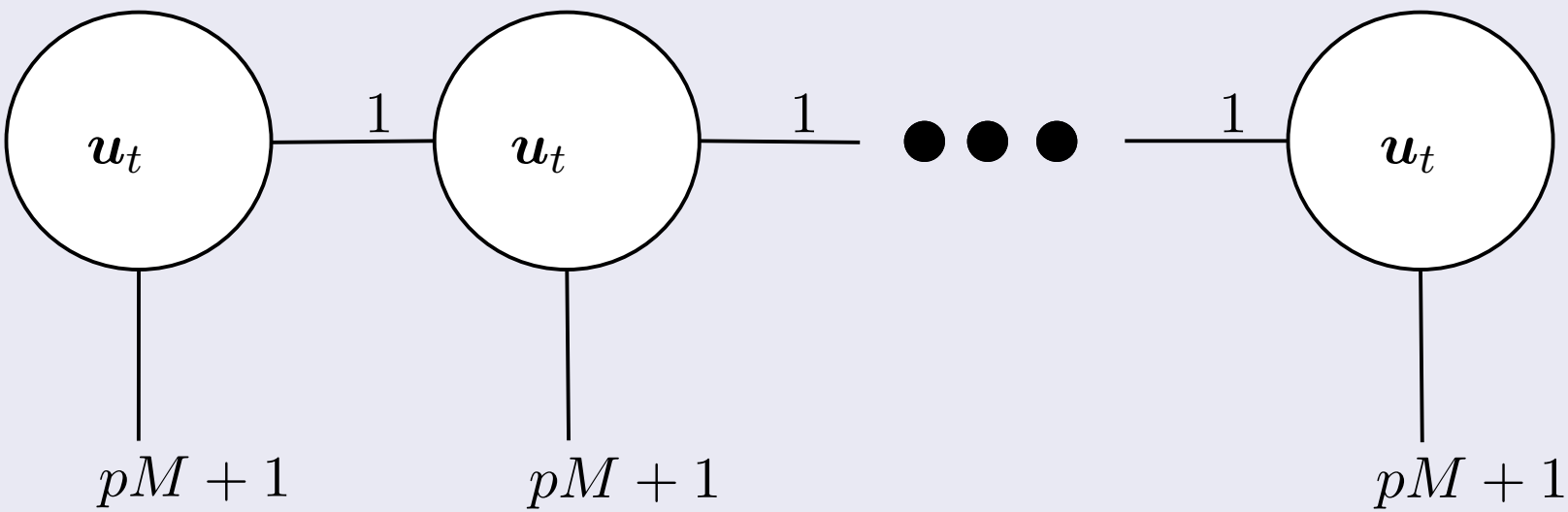
$$\boldsymbol{u}_t \circ \boldsymbol{u}_t = \begin{pmatrix} 1 \\ u(t) \\ u(t-1) \end{pmatrix} \begin{pmatrix} 1 & u(t) & u(t-1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & u(t) & u(t-1) \\ u(t) & u(t)^2 & u(t)\,u(t-1) \\ u(t-1) & u(t)\,u(t-1) & u(t-1)^2 \end{pmatrix}$$



Symmetric rank-1 input tensor

$$\overbrace{\boldsymbol{u}_t \circ \boldsymbol{u}_t \circ \cdots \circ \boldsymbol{u}_t}^{d} \in \mathbb{R}^{(pM+1)\times\cdots\times(pM+1)}$$



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$$y(t) = f(u(t), u(t-1))$$

$$= \boldsymbol{v}^T \operatorname{vec}(\boldsymbol{u}_t \circ \boldsymbol{u}_t)$$

$$= \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & \cdots & v_9 \end{pmatrix} \begin{pmatrix} 1 \\ u(t) \\ u(t-1) \\ u(t) \\ \vdots \\ u(t-1)^2 \end{pmatrix}$$

MIMO Volterra system

- $\boldsymbol{y}(t) = \boldsymbol{V}\,\mathrm{vec}\left(\boldsymbol{u}_t \circ \boldsymbol{u}_t \circ \cdots \circ \boldsymbol{u}_t\right)$
- $\boldsymbol{V} \in \mathbb{R}^{l \times (pM+1)^d}$ contains coefficients of l polynomials

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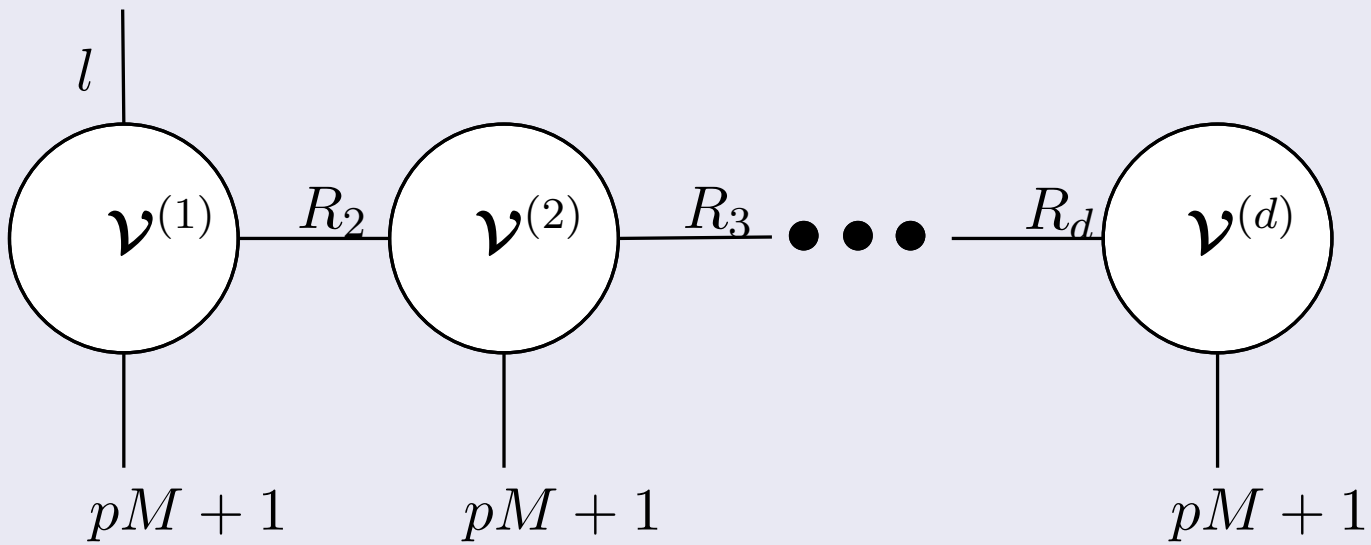
V matrix as a tensor network



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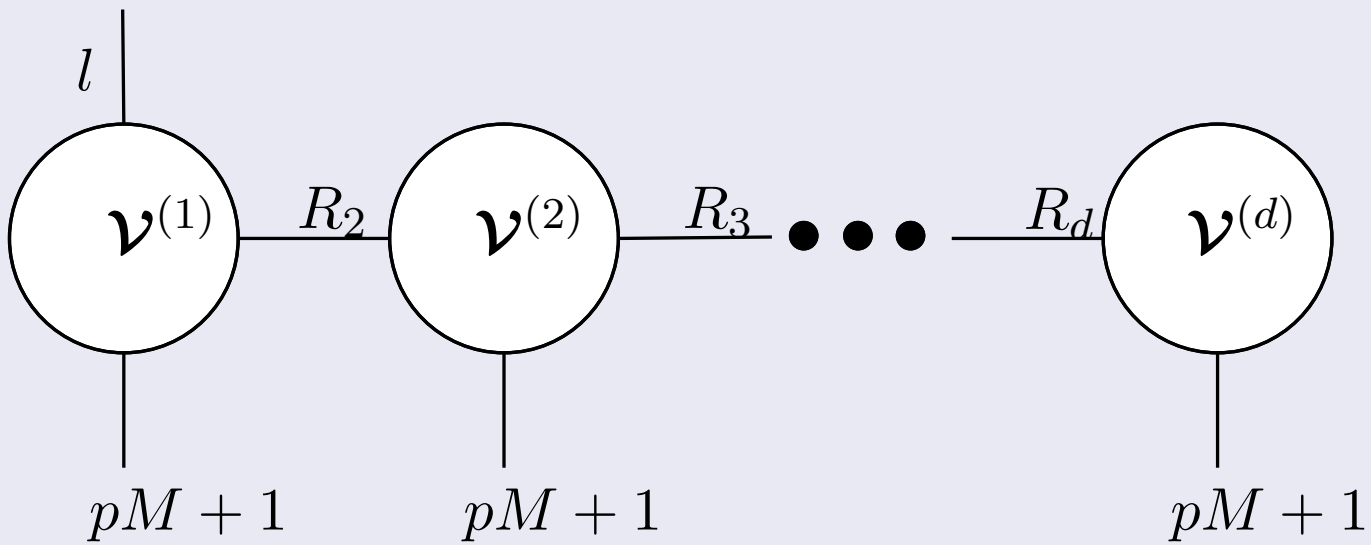
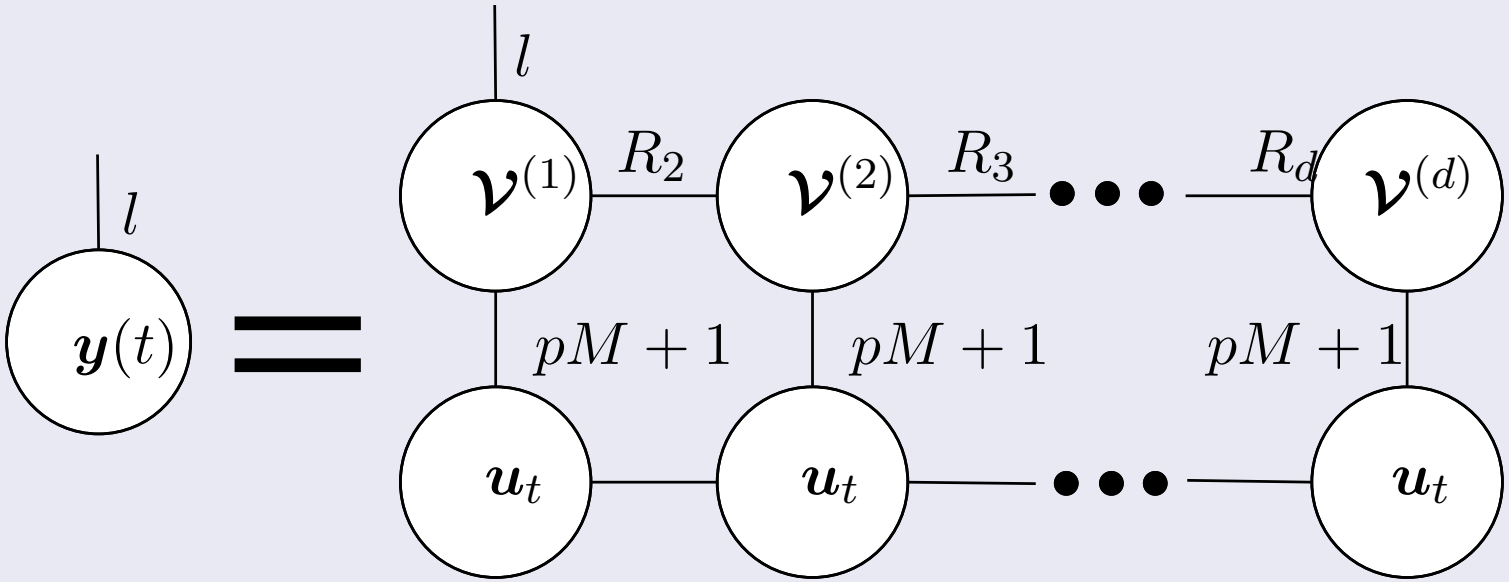


Lifting the curse of dimensionality

Storage of tensor network requires $\approx O(d(pM+1)R^2)$ elements

MIMO Volterra tensor network



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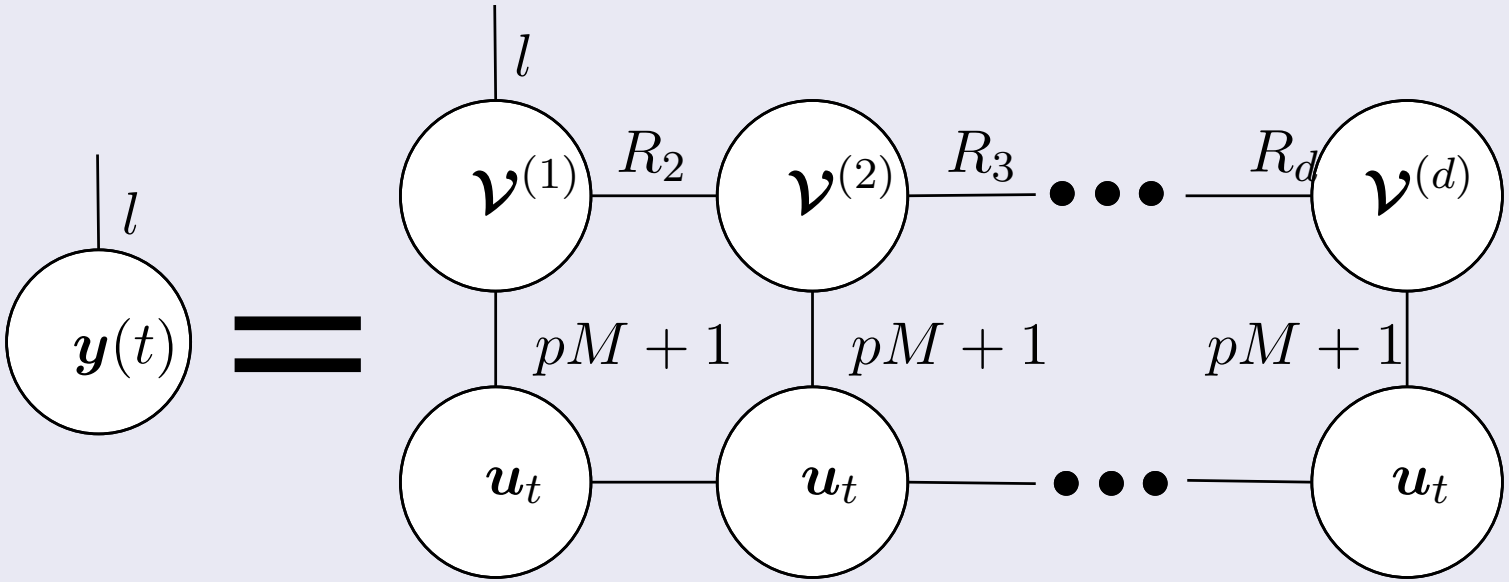

Tensor Network math

$$\sum_{r_2=1}^{R_2} \cdots \sum_{r_d=1}^{R_d} \sum_{i_1=1}^{pn_u+1} \cdots \sum_{i_d=1}^{pn_u+1} \sum_{j_2=1}^{1} \cdots \sum_{j_d=1}^{1} \boldsymbol{\nu}^{(1)}(l, i_1, r_2)\boldsymbol{\nu}^{(2)}(r_2, i_2, r_3)$$

$$\boldsymbol{\nu}^{(d-1)}(r_{d-1}, i_d, r_d)\, \boldsymbol{\nu}^{(d)}(r_d, i_d)\, \boldsymbol{u_t}(i_1, j_2)\, \boldsymbol{u_t}(j_2, i_2, j_3) \cdots \boldsymbol{u_t}(j_d, i_d).$$

MIMO Volterra system identification problem

Given measured inputs $\{\boldsymbol{u}(t) \in \mathbb{R}^p\}_{t=1}^N$, outputs $\{\boldsymbol{y}(t) \in \mathbb{R}^l\}_{t=1}^N$, and an unknown MIMO Volterra model $\boldsymbol{V}$, estimate the tensor network tensors $\boldsymbol{\mathcal{V}}^{(1)}, \boldsymbol{\mathcal{V}}^{(2)}, \ldots, \boldsymbol{\mathcal{V}}^{(d)}$.

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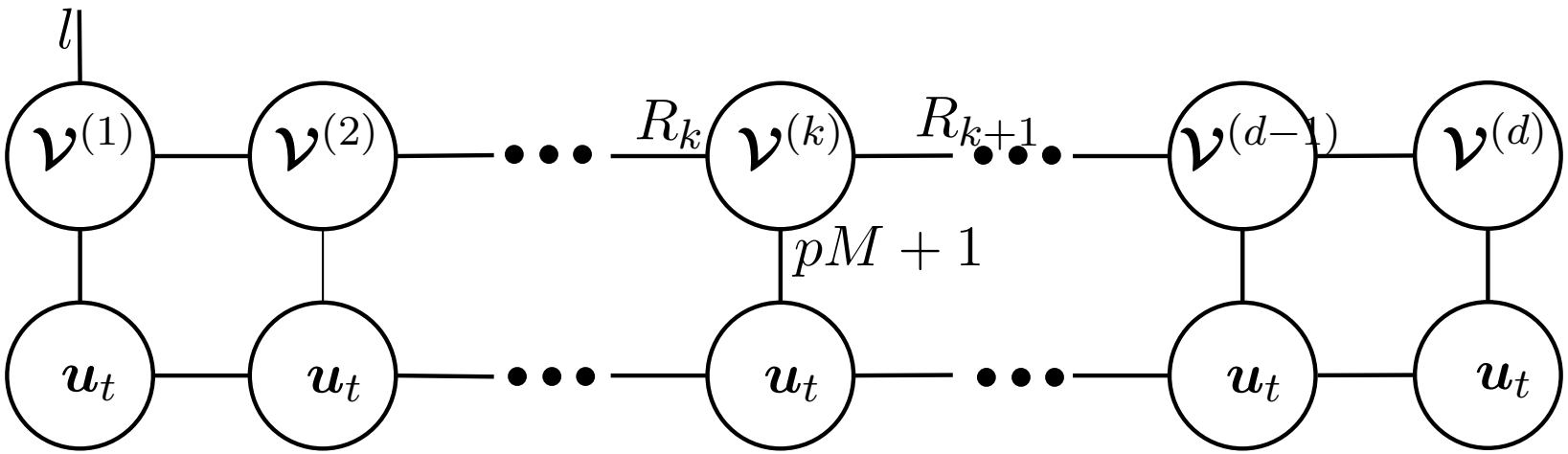
Alternating Linear Scheme (ALS)

- initialize unknown tensors $\boldsymbol{\mathcal{V}}^{(1)}, \boldsymbol{\mathcal{V}}^{(2)}, \ldots, \boldsymbol{\mathcal{V}}^{(d)}$
- for $1 \leq k \leq d$
  - Assume $\boldsymbol{\mathcal{V}}^{(1)}, \ldots, \boldsymbol{\mathcal{V}}^{(k-1)}, \boldsymbol{\mathcal{V}}^{(k+1)}, \ldots, \boldsymbol{\mathcal{V}}^{(d)}$ known and update $\boldsymbol{\mathcal{V}}^{(k)}$.
  - Update $\boldsymbol{\mathcal{V}}^{(k)} \Rightarrow$ solve small linear system for $\mathrm{vec}(\boldsymbol{\mathcal{V}}^{(k)})$


l
$\boldsymbol{\nu}^{(1)}$
$\boldsymbol{\nu}^{(2)}$
R_k
$\boldsymbol{\nu}^{(k)}$
R_{k+1}
$\boldsymbol{\nu}^{(d-1)}$
$\boldsymbol{\nu}^{(d)}$
$pM+1$
$\boldsymbol{u}_t$
$\boldsymbol{u}_t$
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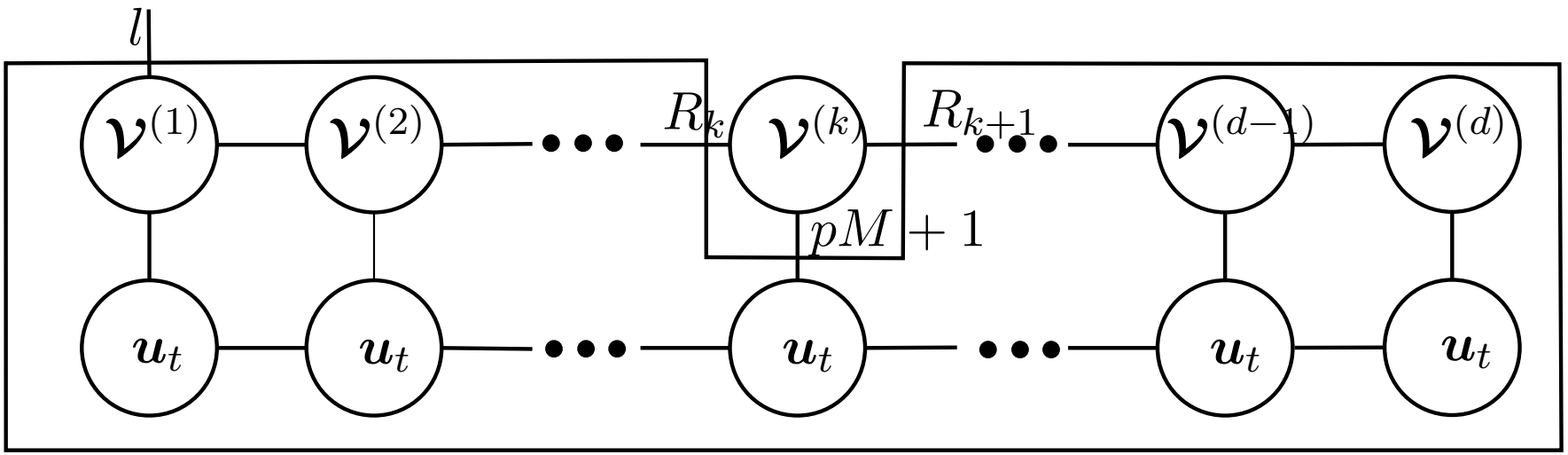
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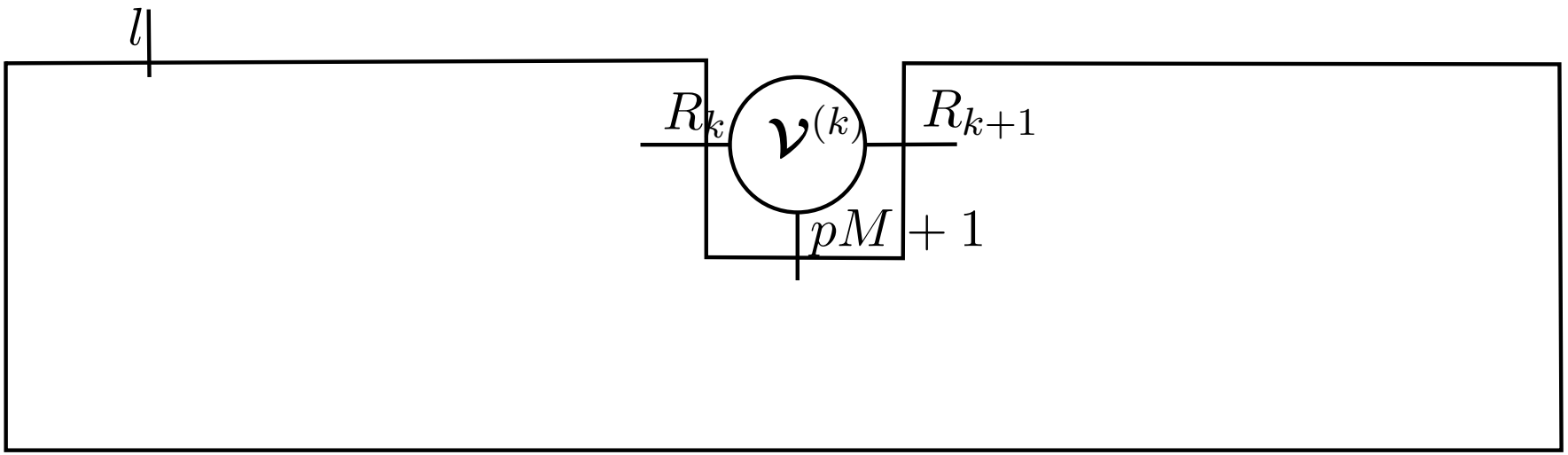
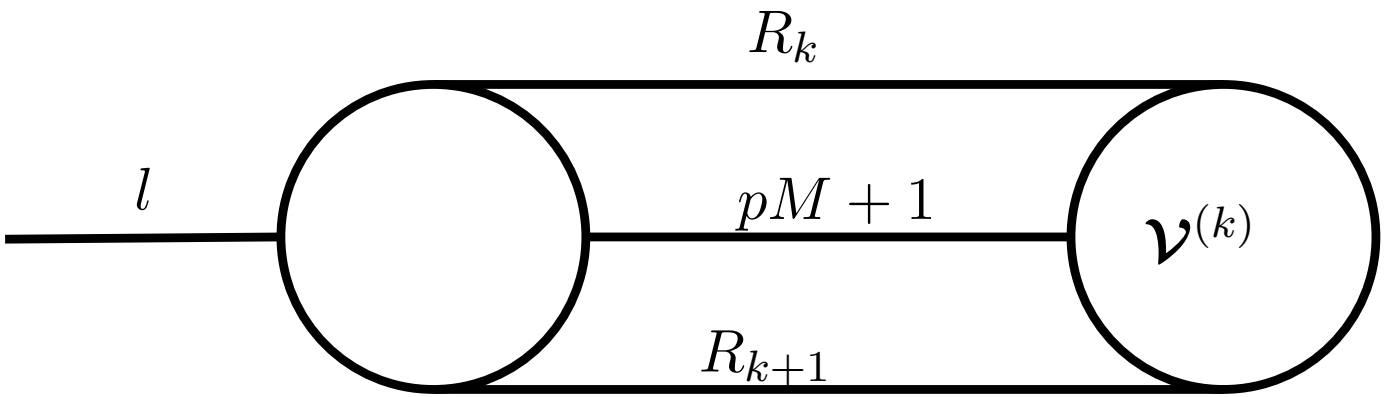

l
R_k
$\boldsymbol{\nu}^{(k)}$
R_{k+1}
$pM + 1$


R_k
l
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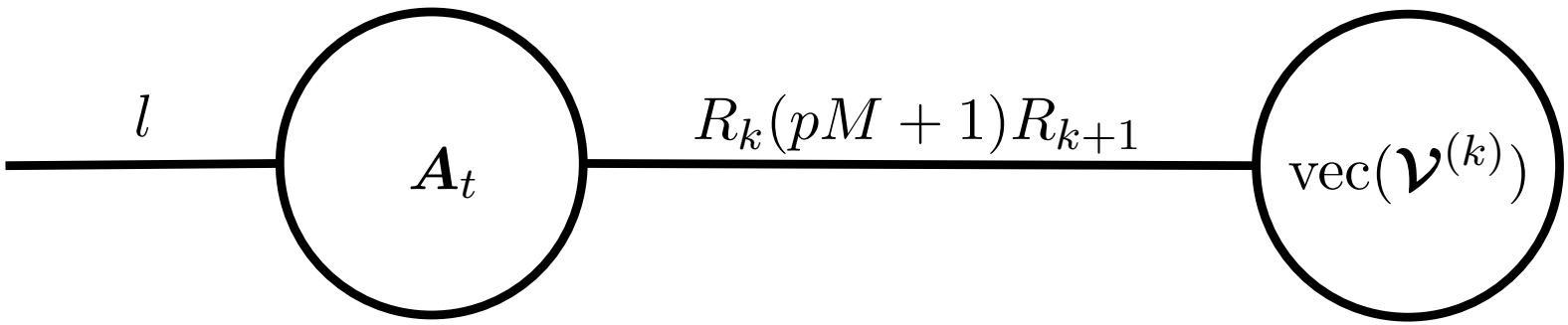
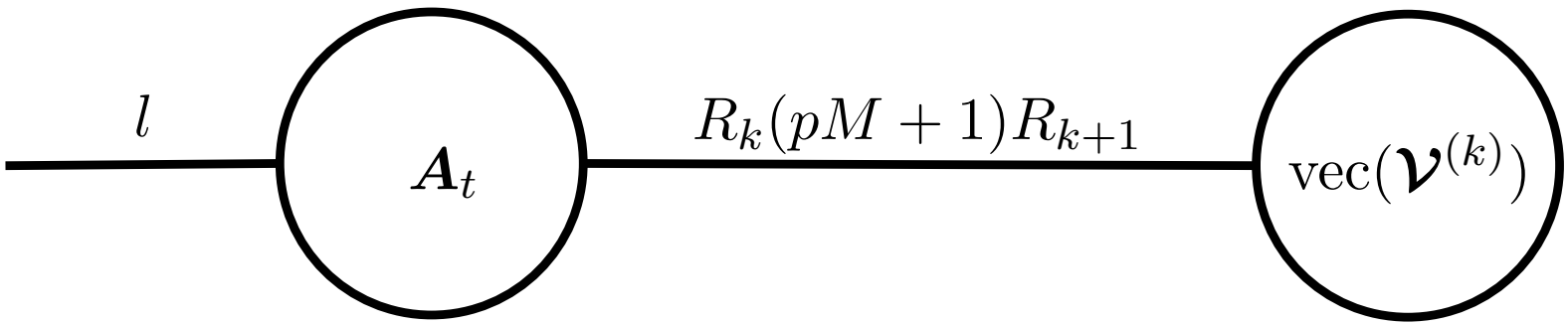

l
$\boldsymbol{A}_t$
$R_k(pM+1)R_{k+1}$
$\mathrm{vec}(\boldsymbol{\mathcal{V}}^{(k)})$



$$\boldsymbol{y}(t) = \boldsymbol{A}_t \ \text{vec}(\boldsymbol{\mathcal{V}}^{(k)})$$

Updating $\boldsymbol{\mathcal{V}}^{(k)}$

$$\begin{pmatrix} \boldsymbol{y}(1) \\ \boldsymbol{y}(2) \\ \vdots \\ \boldsymbol{y}(N) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{A}_N \end{pmatrix} \operatorname{vec}(\boldsymbol{\mathcal{V}}^{(k)})$$

MIMO Volterra ALS system identification algorithm

- Initialize unknown tensors $\boldsymbol{\mathcal{V}}^{(1)}, \boldsymbol{\mathcal{V}}^{(2)}, \ldots, \boldsymbol{\mathcal{V}}^{(d)}$
- While stopping criterion not satisfied
  - For $k = 1 : d$, Update $\boldsymbol{\mathcal{V}}^{(k)}$
  - For $k = d - 1 : -1 : 2$, Update $\boldsymbol{\mathcal{V}}^{(k)}$
- End while

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Remarks

- Stopping criterion: fixed number of updates, residual on $||\boldsymbol{Y} - \hat{\boldsymbol{Y}}||_F / ||\boldsymbol{Y}||_F$, ...
- How to choose $R_2, \ldots, R_d$? ($\Rightarrow$ Modified ALS updates the ranks automatically)
- Orthogonalization step after solving the linear system ensures numerical stability.

Numerical Experiment

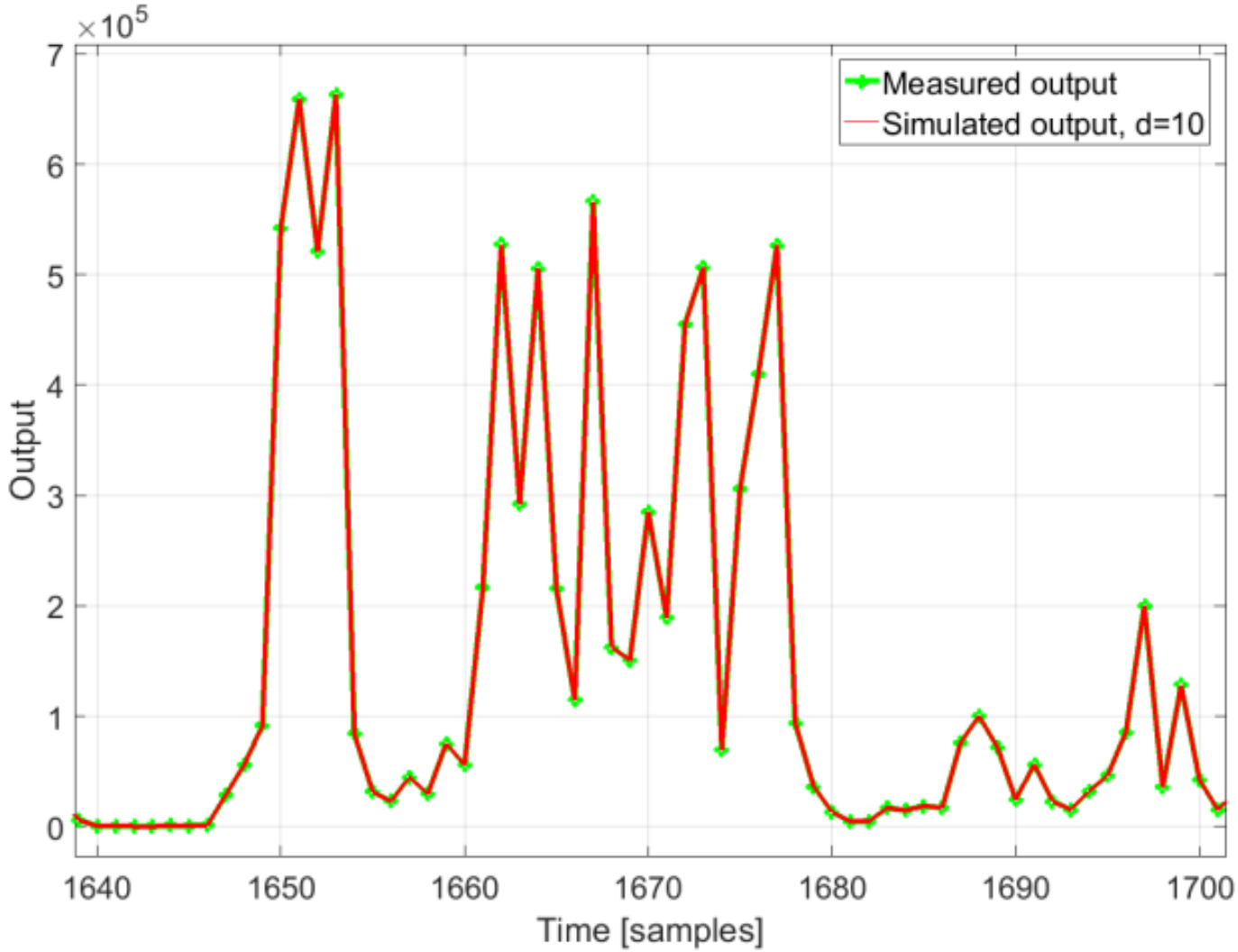
- Generated Volterra system with $d = 10$ and $M = 7$.
- The ith symmetric Volterra kernel contains entries

$$h_i(k_1, \ldots, k_i) = e^{-.1 \sum_{j=1}^{i} k_j^2}.$$

- white noise input: 700 samples for identification, 2000 samples for validation.
- relative approximation error $||\boldsymbol{Y} - \hat{\boldsymbol{Y}}||/||\boldsymbol{Y}|| \leq 10^{-4}$.

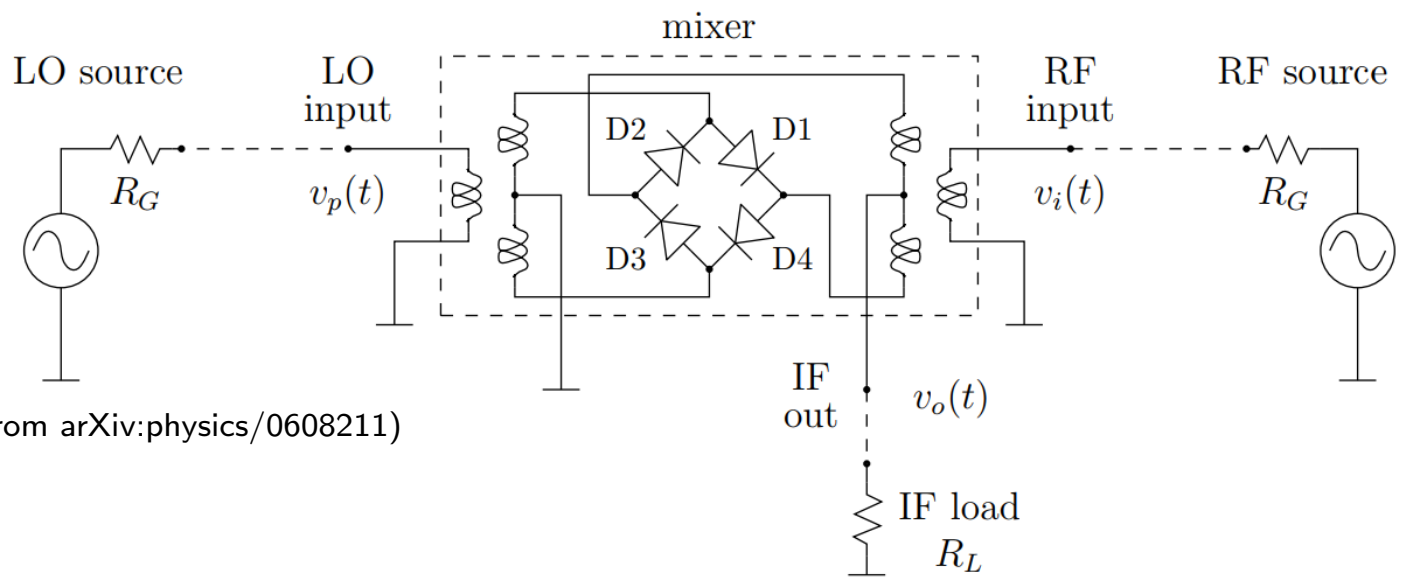
Table 1
Run times, maximal TT-rank and number of estimated Volterra tensor elements for an increasing degree d.

d	Run time [seconds]			max TT-rank	$(M+1)^d$
	$\boldsymbol{U}^\dagger \boldsymbol{y}$	ALS	MALS		
2	0.017	0.240	0.077	6	64
3	0.253	0.223	0.251	8	512
4	39.36	1.771	0.415	8	4096
5	NA	4.288	0.587	8	32768
6	NA	3.065	0.791	8	262144
7	NA	6.783	0.961	8	2097152
8	NA	13.11	1.199	8	16777216
9	NA	14.42	1.384	8	134217728
10	NA	18.37	1.576	8	1.0737e9


×10^5
7
6
5
4
3
2
1
0
Output
1640
1650
1660
1670
1680
1690
1700
Time [samples]
Measured output
Simulated output, d=10

Experiment: double balanced mixer for upconversion

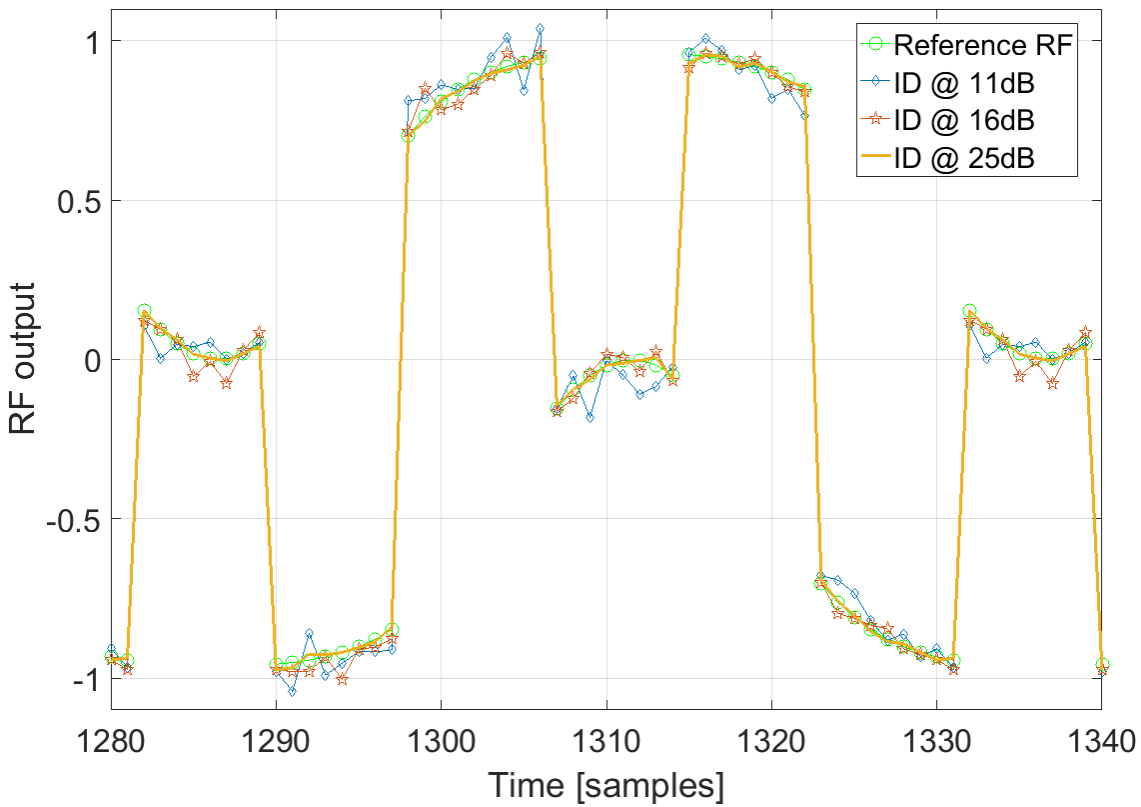
- 2 inputs: LO = 100Hz sine, IF= 300Hz square-wave.
- output= RF.
- Sampling frequency: 5kHz for 1 second
- Five different noise levels added to RF output: SNR=11dB,13dB,16dB,19dB,25dB.



(image from arXiv:physics/0608211)

Experiment

- $M = 2, d = 11 \Rightarrow (pM + 1)^d = 5^{11} = 48828125$
- $R_2 = \cdots = R_{11} = 5 \Rightarrow$ Each TN node has 125 unknowns
- Identification takes around 2 seconds on standard desktop pc



Time-varying MIMO Volterra identification

$$\boldsymbol{V}(t+1)^T = \boldsymbol{V}(t)^T + \Delta\boldsymbol{V}(t)^T$$
$$\boldsymbol{y}(t)^T = \mathrm{vec}(\boldsymbol{u}_t \circ \cdots \circ \boldsymbol{u}_t)^T \, \boldsymbol{V}(t)^T + \boldsymbol{e}(t)$$

- Time-varying linear state space
- "State" is a matrix $\boldsymbol{V}^T \in \mathbb{R}^{(pM+1)^d \times l}$
- Tensor network Kalman filter for the recursive estimation of $\boldsymbol{V}$

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"Polynomial in the input" state space

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,\mathrm{vec}(\boldsymbol{u}_t \circ \cdots \circ \boldsymbol{u}_t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\,\boldsymbol{x}(t) + \boldsymbol{D}\,\mathrm{vec}(\boldsymbol{u}_t \circ \cdots \circ \boldsymbol{u}_t).$$

- MOESP subspace identification with Tensor Networks

Open source MATLAB/Octave implementations

`https://github.com/kbatseli/`

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